Random Branched Covers of the Torus: An Exercise in Statistical Topology

Abstract

What is the expected Euler characteristic of random branched cover of the Torus? This quantum gravity problem motivates a number of statistical problems in group theory and topology. We survey the techniques involved.

1 Square-Tiled Surfaces

1.1 Hurwitz Encoding

We can build a surface by gluing together n squares along parallel sides. The identification is encoded by two permutations, $V, H \in S_n$. These two permutations determine the surface uniquely and vice versa.

Let's check this again using some algebraic topology. A $\square$-tiled surface is a covering space of the once-punctured torus. These covering spaces can be identified with homomorphisms from $\pi_1(\square) \to S_n$. Loops on the base lift to paths in the cover[1]. The once-punctured torus is contractible to a wedge of two circles, so $\pi_1(\square) = \mathbb{F}_2 = \langle V, H \rangle$ the free group on two generators. Homomorphisms are determined by the images of V, H in S_n.

For reference, the fundamental group of the torus is $\langle x, y | xyx^{-1}y^{-1} \rangle$ since the boundary path of the square is contractible. Branched covers of the torus are indexed by commuting permutations, $x, y \in S_n$ with $xy = yx$.

Our main question: what is the expected Euler characteristic of a random branched cover of the once-punctured torus. There are two similar questions depending on whether or not we count branched covers up to conjugacy. If conjugate surfaces are equivalent $(V, H) \sim (gVg^{-1}, gHg^{-1})$ the number of surfaces can be arranged into a quasimodular function. If not, we are asking about the behavior of commutators $VHV^{-1}H^{-1}$ with V, H chosen randomly in S_n.

If we want the Euler characteristic of the branched cover, we need n copes of the torus (which contribute 0) and count the degree of the ramification points. That can be made rigorous using integration with respect to Euler characteristic.

$$\chi(C) = n\chi(\mathbb{T}^2) + \sum_{ram} (e_P - 1) = n - |VHV^{-1}H^{-1}|$$

Here $|\sigma|$ is the number of cycles of σ for $\sigma \in S_n$.

[1] A covering space is also a fiber bundle whose fibers are finite sets.

1.2 Ewens Measure and Representation Theory

After a few dead ends, almost a solution. Let's calculate a generating function

$$\frac{1}{n!^2} \sum_{g,h \in S_n} \theta^{|ghg^{-1}h^{-1}|}$$

The number of cycles $|\sigma|$ is a class function, so $\theta^|\sigma|$ can also be expanded into a sum of characters.

$$\theta^{|\sigma|} = \sum_{\lambda} a_\lambda \langle \lambda | \sigma \rangle$$

Using some tricks I found in a physics paper, we can compute the average over the commutators.

$$\frac{1}{n!^2} \sum_{g,h \in S_n} \langle \lambda | ghg^{-1}h^{-1} \rangle = \frac{1}{\langle \lambda | 1 \rangle} = \frac{1}{\dim \lambda}$$

Had been stuck on how to compute the coefficients, a_λ.

Gnedin, Gorin and Kerov talk about block characters of the permutation group, [3].

- Depend only on the number of cycles $\ell_n(g)$.
- They are class functions. $\chi(ab) = \chi(ba)$
- $\chi(g_i g_j^{-1})$ is a hermitian matrix with non-negative eigenvalues.

They show that $k^{\ell_n(g)}$ is character of the S_n group acting on words of length n with letters in $\{1, 2, \ldots, k\}$. We can extrapolate to $\theta^{|\ell(g)|}$ with $\theta > 0$.

With a clever use of Schur-Weyl duality they show

$$\theta^{|\sigma|} = \sum_{\lambda} s_\theta(\lambda) \langle \lambda | \sigma \rangle$$

where $s_k(\lambda)$ is the number of *semistandard* tableau of shape λ with letters in $\{1, 2, \ldots, k\}$.

$$s_k(\lambda) = \prod_{\square \in \lambda} \frac{\theta + c_\square}{h_\square}$$

This is an nth degree polynomial in k which can be interpolated to a real-valued function in θ. We are now ready to derive our generating function.

$$\begin{aligned}
\frac{1}{n!^2} \sum \theta^{|ghg^{-1}h^{-1}|} &= \sum_{\lambda} \frac{s_k(\lambda)}{\dim \lambda} \\
&= \sum_{\lambda} \frac{\prod_{\square \in \lambda} \frac{\theta + c(\square)}{h(\square)}}{\frac{n!}{\prod_{\square \in \lambda} h(\square)}} \\
&= \frac{1}{n!} \sum_{\lambda} \prod_{\square \in \lambda} (\theta + c(\square))
\end{aligned}$$

This formula is exercise 7.68(e) in Enumerative Combinatorics by Richard Stanley. In that exercise it's shown the expected number of k-cycles of $ghg^{-1}h^{-1}$ is $\frac{n}{k}$ plus an error term. Therefore the expected number of cycles is the Harmonic series in the large n limit: $nH_n = n\left(1 + \frac{1}{2} + \frac{1}{3} + \dots\right)$. Alexandru Nica shows the number of k-cycles in $ghg^{-1}h^{-1}$ tends to the Poisson distribution as $n \to \infty$.

To summarize the expected Euler characteristic is $\mathbb{E}(\chi) \approx n - nH_n = -n(\frac{1}{2} + \frac{1}{3} + \dots)$.

References

[1] "MathOverflow: Hurwitz Encoding" http://mathoverflow.net/questions/9415/hurwitz-encoding

[2] hep-th/9411210 Lectures on 2D Yang-Mills Theory, Equivariant Cohomology and Topological Field Theories. Stefan Cordes, Gregory Moore, Sanjaye Ramgoolam

[3] arXiv:1108.5044 Block characters of the symmetric groups. Alexander Gnedin, Vadim Gorin, Sergei Kerov.

[4] Alexandru Nica. "On the number of cycles of given length of a free word in several random permutations" Random Structures & Algorithms Volume 5, Issue 5, pages 703730, December 1994